Differential Geometry

MMT-C204

Module-3

Fundamental Theorem of Surfaces in $\mathbb{R}^3$

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3. Fundamental Theorem of Surfaces in $\mathbb{R}^3$

3.1. Frobenius Theorem.

Let $\mathcal{O}$ be an open subset of $\mathbb{R}^2$, $f, g : \mathcal{O} \to \mathbb{R}$ smooth maps, $(x_0, y_0) \in \mathcal{O}$, and $c_0 \in \mathbb{R}$. Then the initial value problem for the following ODE system,

\[
\begin{align*}
\frac{\partial u}{\partial x} &= f(x, y), \\
\frac{\partial u}{\partial y} &= g(x, y), \\
u(x_0, y_0) &= c_0,
\end{align*}
\]

has a smooth solution defined in some disk centered at $(x_0, y_0)$ for any given $(x_0, y_0) \in \mathcal{O}$ if and only if $f, g$ satisfy the compatibility condition

\[
\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}
\]

in $\mathcal{O}$. Moreover, we can use integration to find the solution as follows: Suppose $u(x, y)$ is a solution, then the Fundamental Theorem of Calculus implies that

\[
u(x, y) = v(y) + \int_{x_0}^{x} f(t, y) \, dt
\]

for some $v(y)$ such that $v(y_0) = c_0$. 
But
\[ u_y = v'(y) + \int_{x_0}^{x} \frac{\partial f}{\partial y} \, dt = v'(y) + \int_{x_0}^{x} \frac{\partial g}{\partial x} \, dx = v'(y) + g(x, y) - g(x_0, y) \]
should be equal to \( g(x, y) \), so \( v'(y) = g(x_0, y) \). But \( v(y_0) = c_0 \), hence
\[ v(y) = c_0 + \int_{y_0}^{y} g(x_0, s) \, ds. \]
In other words, the solution for the initial value problem is
\[ u(x, y) = c_0 + \int_{y_0}^{y} g(x_0, s) \, ds + \int_{x_0}^{x} f(t, y) \, dt. \]

Given smooth maps \( A, B : \mathcal{O} \times \mathbb{R} \to \mathbb{R} \), we now consider the following first order PDE system for \( u : \mathcal{O} \to \mathbb{R} \):
\[
\begin{cases}
\frac{\partial u}{\partial x} = A(x, y, u(x, y)), \\
\frac{\partial u}{\partial y} = B(x, y, u(x, y)).
\end{cases}
\]
If we have a smooth solution \( u \) for (3.1.1), then \( (u_x)_y = (u_y)_x \). But
\[
(u_x)_y = (A(x, y, u(x, y)))_y = A_y + A_u u_y = A_y + A_u B \\
= (u_y)_x = (B(x, y, u(x, y)))_x = B_x + B_u u_x = B_x + B_u A.
\]
Thus \( A, B \) must satisfy the following condition:
\[
A_y + A_u B = B_x + B_u A. 
\]
The Frobenius Theorem states that (3.1.2) is both a necessary and sufficient condition for the first order PDE system (3.1.1) to be solvable. We will see from the proof of this theorem that, although we are dealing with a PDE, the algorithm to construct solutions of this PDE is to first solve an ODE system in \( x \) variable (first equation of (3.1.1) on the line \( y = y_0 \)), and then solve a family of ODE systems in \( y \) variables (the second equation of (3.1.1) for each \( x \)). The condition (3.1.2) guarantees that this process produces a solution of (3.1.1). We state the Theorem for \( u : \mathcal{O} \to \mathbb{R}^n \):

**Theorem 3.1.1. (Frobenius Theorem)** Let \( U_1 \subset \mathbb{R}^2 \) and \( U_2 \subset \mathbb{R}^n \) be open subsets, \( A = (A_1, \ldots, A_n), B = (B_1, \ldots, B_n) : U_1 \times U_2 \to \mathbb{R}^n \) smooth maps, \( (x_0, y_0) \in U_1 \), and \( p_0 \in U_2 \). Then the following first order system
\[
\begin{cases}
\frac{\partial u}{\partial x} = A(x, y, u(x, y)), \\
\frac{\partial u}{\partial y} = B(x, y, u(x, y)), \\
u(x_0, y_0) = p_0,
\end{cases}
\]
has a smooth solution for \( u \) in a neighborhood of \( (x_0, y_0) \) for all possible \( (x_0, y_0) \in U_1 \) and \( p_0 \in U_2 \) if and only if
\[
(A_i)_y + \sum_{j=1}^{n} \frac{\partial A_i}{\partial u_j} B_j = (B_i)_x + \sum_{j=1}^{n} \frac{\partial B_i}{\partial u_j} A_j, \quad 1 \leq i \leq n,
\]
hold identically on \( U_1 \times U_2 \).
Equation (3.1.3) written in coordinates gives the following system:

\[
\begin{align*}
\frac{\partial u_i}{\partial x} &= A_i(x, y, u_1(x, y), \ldots, u_n(x, y)), \quad 1 \leq i \leq n, \\
\frac{\partial u_i}{\partial y} &= B_i(x, y, u_1(x, y), \ldots, u_n(x, y)), \quad 1 \leq i \leq n, \\
u_i(x_0, y_0) &= p^0_i,
\end{align*}
\]

where \( p_0 = (p^0_1, \ldots, p^0_n) \).

We call (3.1.4) the compatibility condition for the first order PDE (3.1.3).

To prove the Frobenius Theorem we need to solve a family of ODEs depending smoothly on a parameter, and we need to know whether the solutions depend smoothly on the initial data and the parameter. This was answered by the following Theorem in ODE:

**Theorem 3.1.2.** Let \( O \) be an open subset of \( \mathbb{R}^n \), \( t_0 \in (a_0, b_0) \), and \( f : [a_0, b_0] \times O \times [a_1, b_1] \to \mathbb{R}^n \) a smooth map. Given \( p \in O \) and \( r \in [a_1, b_1] \), let \( y^{p,r} \) denote the solution of

\[
\frac{dy}{dt} = f(t, y(t), r), \quad y(t_0) = p,
\]

and \( u(t, p, r) = y^{p,r}(t) \). Then \( u \) is smooth in \( t, p, r \).

**Proof. Proof of Frobenius Theorem**

If \( u = (u_1, \ldots, u_n) \) is a smooth solution of (3.1.3), then \( \frac{\partial}{\partial y} \frac{\partial u_i}{\partial x} = \frac{\partial}{\partial x} \frac{\partial u_i}{\partial y} \). Use the chain rule to get

\[
\begin{align*}
\frac{\partial}{\partial y} \frac{\partial u_i}{\partial x} &= \frac{\partial}{\partial y} A_i(x, y, u_1(x, y), \ldots, u_n(x, y)) = \frac{\partial A_i}{\partial y} + \sum_{j=1}^{n} \frac{\partial A_i}{\partial u_j} \frac{\partial u_j}{\partial y} \\
&= \frac{\partial A_i}{\partial y} + \sum_{j=1}^{n} \frac{\partial A_i}{\partial u_j} B_j, \\
\frac{\partial}{\partial x} \frac{\partial u_i}{\partial y} &= \frac{\partial}{\partial x} B_i(x, y, u(x, y)) = \frac{\partial B_i}{\partial x} + \sum_{j=1}^{n} \frac{\partial B_i}{\partial u_j} \frac{\partial u_j}{\partial x} \\
&= \frac{\partial B_i}{\partial x} + \sum_{j=1}^{n} \frac{\partial B_i}{\partial u_j} A_j,
\end{align*}
\]

so the compatibility condition (3.1.4) must hold.

Conversely, assume \( A, B \) satisfy (3.1.4). To solve (3.1.3), we proceed as follows: The existence and uniqueness Theorem of solutions of ODE implies that there exist \( \delta > 0 \) and \( \alpha : (x_0 - \delta, x_0 + \delta) \to U_2 \) satisfying

\[
(3.1.5) \quad \begin{cases} 
\frac{d\alpha}{dx} = A(x, y_0, \alpha(x)), \\
\alpha(x_0) = p_0.
\end{cases}
\]

For each fixed \( x \in (x_0 - \delta, x_0 + \delta) \), let \( \beta^x(y) \) denote the unique solution of the following ODE in \( y \) variable:

\[
(3.1.6) \quad \begin{cases} 
\frac{d\beta^x}{dy} = B(x, y, \beta^x(y)), \\
\beta^x(y_0) = \alpha(x).
\end{cases}
\]

Set \( u(x, y) = \beta^x(y) \). Note that (3.1.6) is a family of ODEs in \( y \) variable depending on the parameter \( x \) and \( B \) is smooth, so by Theorem 3.1.2, \( u \) is smooth in \( x, y \).
Hence \((u_x)_y = (u_y)_x\). By construction, \(u\) satisfies the second equation of (3.1.3) and \(u(x_0, y_0) = p_0\). It remains to prove \(u\) satisfies the first equation of (3.1.3). We will only prove this for the case \(n = 1\), and the proof for general \(n\) is similar. First let
\[
 z(x, y) = u_x - A(x, y, u(x, y)).
\]
But
\[
 z_y = (u_x - A(x, y, u))_y = u_{xy} - A_y - A_u u_y = (u_y)_x - (A_y + A_u B)
\]
\[
 = (B(x, y, u))_x - (A_y + A_u B) = B_x + B_u u_x - (A_y + A_u B)
\]
\[
 = B_x + B_u u_x - (B_x + B_u A) = B_u (u_x - A) = B_u (x, y, u(x, y)) z.
\]
This proves that for each \(x\), \(h^x(y) = z(x, y)\) is a solution of the following differential equation:

\[
(3.1.7)
\frac{dh}{dy} = B_u(x, y, u(x, y)) h.
\]
Since \(\alpha\) satisfies (3.1.5),
\[
 z(x, y_0) = u_x(x, y_0) - A(x, y_0, u(x, y_0)) = \alpha'(x) - A(x, y_0, \alpha(x)) = 0.
\]
So \(h^x\) is the solution of (3.1.7) with initial data \(h^x(y_0) = 0\). We observe that the zero function is also a solution of (3.1.7) with 0 initial data, so by the uniqueness of solutions of ODE we have \(h^x = 0\), i.e., \(z(x, y) = 0\), hence \(u\) satisfies the second equation of (3.1.3).

**Remark 3.1.3.** The proof of Theorem 3.1.1 gives the following algorithm to construct numerical solution of (3.1.3):

1. Solve the ODE (3.1.5) on the horizontal line \(y = y_0\) by a numerical method (for example Runge-Kutta) to get \(u(x_k, y_0)\) for \(x_k = x_0 + k\epsilon\) where \(\epsilon\) is the step size in the numerical method.
2. Solve the ODE system (3.1.6) on the vertical line \(x = x_k\) for each \(k\) to get the value \(u(x_k, y_m)\).

If \(A\) and \(B\) satisfies the compatibility condition, then \(u\) solves (3.1.3).

Let \(gl(n)\) denote the space of \(n \times n\) real matrices. Note that \(gl(n)\) can be identified as \(\mathbb{R}^{n^2}\). For \(P, Q \in gl(n)\), let \([P, Q]\) denote the commutator (also called the bracket) of \(P\) and \(Q\) defined by
\[
[P, Q] = PQ - QP.
\]

**Corollary 3.1.4.** Let \(U\) be an open subset of \(\mathbb{R}^2\), \((x_0, y_0) \in U\), \(C \in gl(n)\), and \(P, Q : U \to gl(n)\) smooth maps. Then the following initial value problem for \(u : U \to gl(n)\)

\[
(3.1.8)
\begin{align*}
 u_x &= u(x, y)P(x, y), \\
 u_y &= u(x, y)Q(x, y), \\
 u(x_0, y_0) &= C
\end{align*}
\]

has a smooth solution \(u\) defined in some small disk centered at \((x_0, y_0)\) for all possible \((x_0, y_0)\) in \(U\) and \(C \in gl(n)\) if and only if

\[
(3.1.9)
P_y - Q_x = [P, Q].
\]
Example 3.1.7. Given \( u, \xi \). This shows that we need for the Fundamental Theorem of surfaces in \( \mathbb{R}^n \). But we observe that the constant map (3.1.8) is a system of 9 equations for 18 functions because (3.1.8) is a system of 3 first order PDE involving six skew symmetric, i.e., \( P^T = -P \) and \( Q^T = -Q \), then \( [P,Q] \) is also skew-symmetric because

\[
[P,Q]^T = (PQ - QP)^T = QTP^T + P^TQ^T = (-Q)(-P) - (-P)(-Q) = \quad PQ - P^TQ = [P,Q].
\]

In this case, equation (3.1.8) becomes a system of 3 first order PDE involving six functions \( p_{12}, p_{13}, p_{23}, q_{12}, q_{13}, q_{23} \).

Proposition 3.1.6. Let \( \mathcal{O} \) be an open subset of \( \mathbb{R}^2 \), and \( P, Q : \mathcal{O} \to gl(n) \) smooth maps such that \( P^T = -P \) and \( Q^T = -Q \). Suppose \( P, Q \) satisfy the compatibility condition (3.1.9), and the initial data \( C \) is an orthogonal matrix. If \( u : \mathcal{O}_0 \to gl(n) \) is the solution of (3.1.8), then \( u(x,y) \) is an orthogonal matrix for all \( (x,y) \in \mathcal{O}_0 \).

Proof. Set

\[
\xi(x,y) = u(x,y)^T u(x,y).
\]

Then \( \xi(x_0,y_0) = u(x_0,y_0)^T u(x_0,y_0) = I \), the identity matrix. Compute directly to get

\[
\xi_x = (u_x)^T u + u^T u_x = (uP)^T u + u^T (uP) = P^T u^T u + u^T uP = P^T \xi + \xi P,
\]

\[
\xi_y = (u_y)^T u + u^T u_y = (uQ)^T u + u^T (uQ) = Q^T u^T u + u^T uQ = Q^T \xi + \xi Q.
\]

This shows that \( \xi \) satisfies

\[
\begin{align*}
\xi_x &= P^T \xi + \xi P, \\
\xi_y &= Q^T \xi + \xi Q, \\
\xi(x_0,y_0) &= I.
\end{align*}
\]

But we observe that the constant map \( \eta(x,y) = I \) is also a solution of the above initial value problem. By the uniqueness part of the Frobenius Theorem, \( \xi = \eta \), so \( u^T u = I \), i.e., \( u(x,y) \) is orthogonal for all \( (x,y) \in \mathcal{O}_0 \).

Next we give some applications of the Frobenius Theorem 3.1.1:

Example 3.1.7. Given \( c_0 > 0 \), consider the following first order PDE

\[
\begin{align*}
\left\{ u_x &= 2 \sin u, \\
u_y &= \frac{1}{2} \sin u, \\
(0,0) &= c_0.
\end{align*}
\]

(3.1.10)
This is system (3.1.3) with \( A(x, y, u) = 2 \sin u, \ B(x, y, u) = \frac{1}{2} \sin u. \) We check the compatibility condition next:

\[
A_y + A_u B = 0 + (2 \cos u)(\frac{1}{2} \sin u) = \cos u \sin u, \\
B_x + B_u A = 0 + (\frac{1}{2} \sin u)(2 \cos u) = \sin u \cos u,
\]

so \( A_y + A_u B = B_x + B_u A. \) Thus by Frobenius Theorem, (3.1.10) is solvable. Next we use the method outlined in the proof of Frobenius Theorem to solve (3.1.10).

(i) The ODE

\[
\begin{aligned}
\frac{d\alpha}{dx} &= 2 \sin \alpha, \\
\alpha(0) &= c_0
\end{aligned}
\]

is separable, i.e., \( \frac{d\alpha}{\sin \alpha} = 2dx, \) so \( \int \frac{d\alpha}{\sin \alpha} = \int 2dx. \) This integration can be solved explicitly:

\[ \alpha(x) = 2 \tan^{-1} \exp(2x + c). \]

But \( \alpha(0) = c_0 = 2 \tan^{-1} e^c \) implies that \( c = \ln(\tan \frac{c_0}{2}). \)

(ii) Solve

\[
\begin{aligned}
\frac{du}{dy} &= \frac{1}{2} \sin u, \\
u(x, 0) &= \alpha(x) = 2 \tan^{-1}(2x + c).
\end{aligned}
\]

We can solve this exactly the same way as in (i) to get

\[ u(x, y) = 2 \tan^{-1}(\exp(2x + y + c)), \]

where \( c = \ln(\tan \frac{c_0}{2}). \)

Moreover, if \( u \) is a solution for (3.1.10), then

\[ (u_x)_y = (2 \sin u)_y = 2 \cos u \ u_y = (2 \cos u)(\frac{1}{2} \sin u) = \cos u \sin u, \]

so \( u \) satisfies the following famous non-linear wave equation, the sine-Gordon equation (or SGE):

\[ u_{xy} = \sin u \cos u. \]

The above example is a special case of the following Theorem of Bäcklund:

**Theorem 3.1.8.** Given a smooth function \( q : \mathbb{R}^2 \to \mathbb{R} \) and a non-zero real constant \( r, \) the following system of first order PDE is solvable for \( u : \mathbb{R}^2 \to \mathbb{R}: \)

\[
\begin{aligned}
&u_x = -q_s + r \sin(u - q), \\
u_t = q_t + \frac{1}{r} \sin(u + q).
\end{aligned}
\]  

(3.1.11)

if and only if \( q \) satisfies the SGE:

\[ q_{st} = \sin q \cos q. \]  

SGE.

Moreover, the solution \( u \) of (3.1.11) is again a solution of the SGE.

**Proof.** If (3.1.11) has a \( C^2 \) solution \( u, \) then the mixed derivatives must be equal. Compute directly to see that

\[
(u_s)_t = -q_s + r \cos(u - q)(u - q)_t \\
= -q_s + r \cos(u - q) \left( \frac{1}{r} \sin(u + q) \right),
\]

and
so we get
(3.1.12) \((u_s)_t = -q_{st} + \cos(u - q) \sin(u + q)\).

A similar computation implies that
(3.1.13) \((u_t)_s = q_{ts} + \cos(u + q) \sin(u - q)\).

Since \(u_{st} = u_{ts}\) and \(\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B\), we get
\[-q_{st} + \cos(u - q) \sin(u + q) = q_{ts} + \cos(u + q) \sin(u - q),\]

so
\[2q_{st} = \sin(u + q) \cos(u - q) - \sin(u - q) \cos(u + q) = \sin(2q) = 2 \sin q \cos q.\]

In other words, the first order PDE (3.1.11) is solvable if and only if \(q\) is a solution of the SGE.

Add (3.1.12) and (3.1.13) to get
\[2u_{st} = \sin(u + q) \cos(u - q) + \sin(u - q) \cos(u + q) = \sin(2u) = 2 \sin u \cos u.\]

This shows that if \(u\) is a solution of (3.1.11) then \(u_{st} = \sin u \cos u. \Box\)

The above theorem says that if we know one solution \(q\) of the SGE then we can solve the first order system (3.1.11) to construct a family of solutions of the SGE (one for each real constant \(r\)). Note that \(q = 0\) is a trivial solution of the SGE. Theorem 3.1.8 implies that system (3.1.11) can be solved for \(u\) with \(q = 0\), i.e.,
(3.1.14) \[
\begin{aligned}
\begin{cases}
  u_s &= r \sin u, \\
  u_t &= \frac{1}{r} \sin u
\end{cases}
\end{aligned}
\]
is solvable. (3.1.14) can be solved exactly the same way as in Example 3.1.7, and we get
\[u(s, t) = 2 \arctan \left( e^{r \frac{s+\frac{t}{2}}{2}} \right),\]
which are solutions of the SGE. The SGE is a so called “soliton” equation, and these solutions are called ”1-solitons”. A special feature of soliton equations is the existence of first order systems that can generate new solutions from an old one.

3.2. Line of curvature coordinates.

**Definition 3.2.1.** A parametrized surface \(f : \mathcal{O} \rightarrow \mathbb{R}^3\) is said to be parametrized by line of curvature coordinates if \(g_{12} = \ell_{12} = 0\), or equivalently, both the first and second fundamental forms are in diagonal forms.

If \(f : \mathcal{O} \rightarrow \mathbb{R}^3\) is a surface parametrized by line of curvature coordinates, then
\[I = g_{11}dx_1^2 + g_{22}dx_2^2, \quad II = \ell_{11}dx_1^2 + \ell_{22}dx_2^2.\]
The principal, Gaussian and mean curvatures are given by the following formulas:
\[k_1 = \frac{\ell_{11}}{g_{11}}, \quad k_2 = \frac{\ell_{22}}{g_{22}}, \quad H = \frac{\ell_{11}}{g_{11}} + \frac{\ell_{22}}{g_{22}}, \quad K = \frac{\ell_{11}\ell_{22}}{g_{11}g_{22}}.\]
**Example 3.2.2.** Let \( u : [a, b] \rightarrow \mathbb{R} \) be a smooth function, and

\[
 f(y, \theta) = (u(y) \sin \theta, y, u(y) \cos \theta),
\]

so the image of \( f \) is the surface of revolution obtained by rotating the curve \( z = u(y) \) in the \( yz \)-plane along the \( y \)-axis. Then

\[
 f_y = (u'(y) \sin \theta, 1, u'(y) \cos \theta), \\
 f_\theta = (u(y) \cos \theta, 0, -u(y) \sin \theta), \\
 N = \frac{f_y \times f_\theta}{|f_y \times f_\theta|} = \frac{(-\sin \theta, u'(y), -\cos \theta)}{\sqrt{1 + (u'(y))^2}}, \\
 f_{yy} = (u''(y) \sin \theta, 0, u''(y) \cos \theta), \\
 f_{y\theta} = (u'(y) \cos \theta, 0, -u'(y) \sin \theta), \\
 f_{\theta\theta} = (-u(y) \sin \theta, 0, -u(y) \cos \theta).
\]

So

\[
 g_{11} = 1 + (u'(y))^2, \quad g_{22} = u(y)^2, \quad g_{12} = 0, \\
 \ell_{11} = \frac{-u''(y)}{\sqrt{1 + (u'(y))^2}}, \quad \ell_{22} = \frac{u(y)}{\sqrt{1 + (u'(y))^2}}, \quad \ell_{12} = 0,
\]

i.e., \((y, \theta)\) is a line of curvature coordinate system.

**Proposition 3.2.3.** If the principal curvatures \( k_1(p_0) \neq k_2(p_0) \) for some \( p_0 \in \mathcal{O} \), then there exists \( \delta > 0 \) such that the ball \( B(p_0, \delta) \) of radius \( \delta \) centered at \( p_0 \) is contained in \( \mathcal{O} \) and \( k_1(p) \neq k_2(p) \) for all \( p \in B(p_0, \epsilon) \).

**Proof.** The Gaussian and mean curvature \( K \) and \( H \) of the parametrized surface \( f : \mathcal{O} \rightarrow \mathbb{R}^3 \) are smooth. The two principal curvatures are roots of \( \lambda^2 - H\lambda + K = 0 \), so we may assume

\[
 k_1 = \frac{H + \sqrt{H^2 - 4K}}{2}, \quad k_2 = \frac{H - \sqrt{H^2 - 4K}}{2}.
\]

Note that the two real roots are distinct if and only if \( u = H^2 - 4K > 0 \). If \( k_1(p_0) \neq k_2(p_0) \), then \( u(p_0) > 0 \). But \( u \) is continuous, so there exists \( \delta > 0 \) such that \( u(p) > 0 \) for all \( p \in B(p_0, \delta) \), thus \( k_1(p) \neq k_2(p) \) in this open disk. \( \square \)

A smooth map \( v : \mathcal{O} \rightarrow \mathbb{R}^3 \) is called a **tangent vector field** of the parametrized surface \( f : \mathcal{O} \rightarrow \mathbb{R}^3 \) if \( v(p) \in Tf_p \) for all \( p \in \mathcal{O} \).

**Proposition 3.2.4.** Let \( f : \mathcal{O} \rightarrow \mathbb{R}^3 \) be a parametrized surface. If its principal curvatures \( k_1(p) \neq k_2(p) \) for all \( p \in \mathcal{O} \), then there exist smooth, o.n., tangent vector fields \( e_1, e_2 \) on \( f \) that are \( e_1(p), e_2(p) \) are eigenvector for the shape operator \( S_p \) for all \( p \in \mathcal{O} \).

**Proof.** This Proposition follows from the facts that

1. self-adjoint operators have an o.n. basis consisting of eigenvectors,
2. the shape operator \( S_p \) is a self-adjoint linear operator from \( Tf_p \) to \( Tf_p \),
3. the formula we gave for self-adjoint operator on a 2-dimensional inner product space implies that the eigenvectors of the shape operator are smooth maps.

\( \square \)
We assume the following Proposition without a proof:

**Proposition 3.2.5.** Let \( f : \mathcal{O} \to \mathbb{R}^3 \) be a parametrized surface. Suppose \( \xi_1, \xi_2 : \mathcal{O} \to \mathbb{R}^3 \) are tangent vector fields of \( f \) such that \( \xi_1(p), \xi_2(p) \) are linearly independent for all \( p \in \mathcal{O} \). Then given any \( p_0 \in \mathcal{O} \) there exists \( \delta > 0 \), an open subset \( U \) of \( \mathbb{R}^2 \), and a diffeomorphism \( \phi : U \to B(p_0, \delta) \) such that \( \xi_1 \) and \( \xi_2 \) are parallel to \( h_{x_1} \) and \( h_{x_2} \) respectively, where \( h = f \circ \phi : U \to \mathbb{R}^3 \).

As a consequence of the above two Propositions, we see that

**Corollary 3.2.6.** Let \( f : \mathcal{O} \to \mathbb{R}^3 \) be a parametrized surface, and \( p_0 \in \mathcal{O} \). If \( k_1(p_0) \neq k_2(p_0) \), then there exist an open subset \( \mathcal{O}_0 \) containing \( p_0 \), an open subset \( U \) of \( \mathbb{R}^2 \), and a diffeomorphism \( \phi : U \to \mathcal{O}_0 \) such that \( f \circ \phi \) is parametrized by lines of curvature coordinates. In other words, we can change coordinates (or reparametrized the surface) near \( p_0 \) by lines of curvature coordinates.

We call \( f(p_0) \) an umbilic point of the parametrized surface \( f : \mathcal{O} \to \mathbb{R}^3 \) if \( k_1(p_0) = k_2(p_0) \). The above Corollary implies that away from umbilic points, we can parametrized a surface by line of curvature coordinates locally.

### 3.3. The Gauss-Codazzi equation in line of curvature coordinates.

Suppose \( f : \mathcal{O} \to \mathbb{R}^3 \) is a surface parametrized by line of curvature coordinates, i.e.,

\[
g_{12} = f_{x_1} \cdot f_{x_2} = 0, \quad \ell_{12} = f_{x_1 x_2} \cdot N = 0.
\]

We define \( A_1, A_2, r_1, r_2 \) as follows:

\[
g_{11} = f_{x_1} \cdot f_{x_1} = A_1^2, \quad g_{22} = f_{x_2} \cdot f_{x_2} = A_2^2, \quad 
\ell_{11} = f_{x_1 x_1} \cdot N = r_1 A_1, \quad \ell_{22} = f_{x_2 x_2} \cdot N = r_2 A_2.
\]

Or equivalently,

\[
A_1 = \sqrt{g_{11}}, \quad A_2 = \sqrt{g_{22}}, \quad r_1 = \frac{\ell_{11}}{A_1}, \quad r_2 = \frac{\ell_{22}}{A_2}.
\]

Set

\[
e_1 = \frac{f_{x_1}}{A_1}, \quad e_2 = \frac{f_{x_2}}{A_2}, \quad e_3 = N.
\]

Then \((e_1, e_2, e_3)\) is an o.n. moving frame on the surface \( f \). Recall that if \( \{e_1, e_2, e_3\} \) is an orthonormal basis of \( \mathbb{R}^3 \), then given any \( \xi \in \mathbb{R}^3 \), \( \xi = \sum_{i=1}^{3} a_i e_i \), where \( a_i = \xi \cdot e_i \). Since \((e_i)_{x_1}\) and \((e_i)_{x_2}\) are vectors in \( \mathbb{R}^3 \), we can write them as linear combinations of \( e_1, e_2 \) and \( e_3 \). We use \( p_{ij} \) to denote the coefficient of \( e_i \) for \((e_j)_{x_1}\), and use \( q_{ij} \) to denote the coefficient of \( e_i \) for \((e_j)_{x_2}\), i.e.,

\[
\begin{align*}
(e_1)_{x_1} &= p_{11} e_1 + p_{21} e_2 + p_{31} e_3, \\
(e_2)_{x_1} &= p_{12} e_1 + p_{22} e_2 + p_{32} e_3, \\
(e_3)_{x_1} &= p_{13} e_1 + p_{23} e_2 + p_{33} e_3, \\
(e_1)_{x_2} &= q_{11} e_1 + q_{21} e_2 + q_{31} e_3, \\
(e_2)_{x_2} &= q_{12} e_1 + q_{22} e_2 + q_{32} e_3, \\
(e_3)_{x_2} &= q_{13} e_1 + q_{23} e_2 + q_{33} e_3,
\end{align*}
\]

where

\[
p_{ij} = (e_j)_{x_1} \cdot e_i, \quad q_{ij} = (e_j)_{x_2} \cdot e_i.
\]
Recall that the matrix $P = (p_{ij})$ and $Q = (q_{ij})$ must be skew-symmetric because

$$
(e_i \cdot e_j)_{x_i} = 0 = (e_i)_{x_i} \cdot e_j + e_i \cdot (e_j)_{x_i} = p_{ji} + p_{ij}.
$$

Next we want to show that $p_{ij}$ and $q_{ij}$ can be written in terms of coefficients of the first and second fundamental forms. We proceed as follows:

$$p_{12} = (e_2)_{x_1} \cdot e_1 = \left( \frac{f_{x_2}}{A_2} \right)_{x_1} \cdot \frac{f_{x_1}}{A_1} = \left( \frac{f_{x_2}}{A_2} - \frac{f_{x_2} (A_2)_{x_1}}{A_2^2} \right) \cdot \frac{f_{x_1}}{A_1}$$

$$= \frac{f_{x_1 x_2}}{A_1 A_2} f_{x_1} - \frac{A_2}{A_1 A_2} f_{x_2} f_{x_1} = \frac{1}{2} \left( f_{x_1} \cdot f_{x_1} \right)_{x_2} = \frac{1}{2} \left( A_2 \right)_{x_2} - 0$$

$$= \frac{A_1 (A_1)_{x_2}}{A_1 A_2} = \frac{(A_1)^2}{A_2}.$$ 

$$p_{31} = (e_1)_{x_1} \cdot e_3 = \left( \frac{f_{x_1}}{A_1} \right)_{x_1} \cdot \frac{e_3}{A_3} = \left( \frac{f_{x_1 x_1}}{A_1} - \frac{f_{x_1} (A_1)_{x_1}}{A_1^2} \right) \cdot N$$

$$= \frac{f_{x_1 x_1}}{A_1} - \frac{(A_1)_{x_1}}{A_1^2} f_{x_1} \cdot N = \ell_{11} \frac{A_1}{A_1} - 0 = r_1.$$ 

$$p_{32} = (e_2)_{x_1} \cdot e_3 = \left( \frac{f_{x_2}}{A_2} \right)_{x_1} \cdot N = \left( \frac{f_{x_2 x_1}}{A_2} - \frac{f_{x_2} (A_2)_{x_1}}{A_2^2} \right) \cdot N = 0$$

So we have proved that

$$p_{12} = \frac{(A_1)^2}{A_2}, \quad p_{31} = r_1, \quad p_{32} = 0.$$ 

In the above computations we have used $f_{x_1} \cdot f_{x_2} = 0, f_{x_1 x_2} \cdot N = 0, f_{x_1} \cdot N = f_{x_2} \cdot N = 0$. Similar computation gives

$$q_{12} = -\frac{(A_2)_{x_1}}{A_1}, \quad q_{31} = 0, \quad q_{32} = r_2.$$ 

Since $P, Q$ are skew-symmetric, we have

$$(3.3.2) \quad P = \begin{pmatrix}
0 & \frac{(A_1)^2}{A_2} & -r_1 \\
-\frac{(A_1)_{x_2}}{A_2} & 0 & 0 \\
r_1 & 0 & 0
\end{pmatrix}, \quad Q = \begin{pmatrix}
0 & -\frac{(A_2)_{x_1}}{A_1} & 0 \\
\frac{(A_2)_{x_2}}{A_1} & 0 & -r_2 \\
r_2 & 0 & 0
\end{pmatrix}.$$ 

So $(3.3.1)$ becomes

$$
\begin{cases}
(e_1)_{x_1} = -(A_1)^2 e_2 + r_1 e_3, \\
(e_2)_{x_1} = (A_1)_{x_2} e_1, \\
(e_3)_{x_1} = -r_1 e_1,
\end{cases}
\begin{cases}
(e_1)_{x_2} = (A_2)_{x_1} e_2, \\
(e_2)_{x_2} = -(A_2)_{x_2} e_1 + r_2 e_3, \\
(e_3)_{x_2} = -r_2 e_2.
\end{cases}
$$

We can also write $(3.3.1)$ in matrix form:

$$(3.3.3) \quad \begin{cases}
(e_1, e_2, e_3)_{x_1} = (e_1, e_2, e_3) P, \\
(e_1, e_2, e_3)_{x_2} = (e_1, e_2, e_3) Q,
\end{cases}$$
where \( P, Q \) are given by (3.3.2). It follows from Corollary 1.04 \( \square \) of the section on Frobenius Theorem \( \square \) that \( P, Q \) must satisfy the compatibility condition

\[ P_{x_2} - Q_{x_1} = PQ - QP. \]

Use the formula of \( P, Q \) given by (3.3.2) to compute directly to get

\[
PQ - QP = \begin{pmatrix} 0 & p & -r_1 \\ -p & 0 & 0 \\ r_1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & q & 0 \\ -q & 0 & -r_2 \\ 0 & r_2 & 0 \end{pmatrix} - \begin{pmatrix} 0 & q & 0 \\ -q & 0 & 0 \\ 0 & r_2 & 0 \end{pmatrix} \begin{pmatrix} 0 & p & -r_1 \\ -p & 0 & 0 \\ r_1 & 0 & 0 \end{pmatrix}.
\]

Then

\[
P_{x_2} - Q_{x_1} = \begin{pmatrix} 0 & p_{x_2} - q_{x_1} & -(r_1)_{x_2} \\ -(p_{x_2} - q_{x_1}) & 0 & (r_2)_{x_1} \\ (r_1)_{x_2} & -(r_2)_{x_1} & 0 \end{pmatrix},
\]

where

\[ p = \frac{(A_1)_{x_2}}{A_2}, \quad q = -\frac{(A_2)_{x_1}}{A_1}. \]

Since

\[ PQ - QP = P_{x_2} - Q_{x_1}, \]

we get

\[
\begin{cases}
\left(\frac{(A_1)_{x_2}}{A_2}\right)_{x_1} + \left(\frac{(A_2)_{x_1}}{A_1}\right)_{x_2} = -r_1 r_2, \\
(r_1)_{x_2} = \frac{(A_1)_{x_2}}{A_2} r_2, \\
(r_2)_{x_1} = -\frac{(A_2)_{x_1}}{A_1} r_1.
\end{cases}
\]

System (3.3.4) is called the Gauss-Codazzi equation. So we have proved:

**Theorem 3.3.1.** Let \( f : \mathcal{O} \to \mathbb{R}^3 \) be a surface parametrized by line of curvature coordinates, and

\[ A_1 = \sqrt{g_{11}}, \quad A_2 = \sqrt{g_{22}}, \quad r_1 = \frac{\ell_{11}}{A_1}, \quad r_2 = \frac{\ell_{22}}{A_2}. \]

Set

\[ e_1 = \frac{f_{x_1}}{A_1}, \quad e_2 = \frac{f_{x_2}}{A_2}, \quad e_3 = \frac{f_{x_1} \times f_{x_2}}{||f_{x_1} \times f_{x_2}||} = e_1 \times e_2. \]

Then \( A_1, A_2, r_1, r_2 \) satisfy the Gauss-Codazzi equation (3.3.4) and

\[
\begin{cases}
(f, e_1, e_2, e_3)_{x_1} = (e_1, e_2, e_3) \begin{pmatrix} A_1 & 0 & \frac{(A_1)_{x_2}}{A_2} & -r_1 \\ 0 & -\frac{(A_1)_{x_2}}{A_2} & 0 & 0 \\ 0 & r_1 & 0 & 0 \\ 0 & 0 & -\frac{(A_2)_{x_1}}{A_1} & 0 \end{pmatrix}, \\
(f, e_1, e_2, e_3)_{x_2} = (e_1, e_2, e_3) \begin{pmatrix} A_2 & 0 & \frac{(A_2)_{x_1}}{A_1} & 0 \\ 0 & -\frac{(A_2)_{x_1}}{A_1} & 0 & -r_2 \\ 0 & 0 & r_2 & 0 \\ 0 & 0 & 0 & r_2 \end{pmatrix}.
\end{cases}
\]
3.4. Fundamental Theorem of surfaces in line of curvature coordinates.

The converse of Theorem 3.3.1 is also true, which is the Fundamental Theorem of surfaces in \( \mathbb{R}^3 \) with respect to line of curvature coordinates:

**Theorem 3.4.1.** Suppose \( A_1, A_2, r_1, r_2 \) are smooth functions from \( \mathcal{O} \) to \( \mathbb{R} \), that satisfy the Gauss-Codazzi equation (3.3.4), and \( A_1 > 0, A_2 > 0 \). Given \( p_0 \in \mathcal{O}, \ y_0 \in \mathbb{R}^3 \), and an o.n. basis \( v_1, v_2, v_3 \) of \( \mathbb{R}^3 \), then there exist an open subset \( \mathcal{O}_0 \) of \( \mathcal{O} \) containing \( p_0 \) and a unique solution \( (f, e_1, e_2, e_3) : \mathcal{O}_0 \to (\mathbb{R}^3)^4 \) of (3.3.5) that satisfies the initial condition

\[
(f, e_1, e_2, e_3)(p_0) = (y_0, v_1, v_2, v_3).
\]

Moreover, \( f \) is a parametrized surface and its first and second fundamental forms are

\[
I = A_1^2 \, dx_1^2 + A_2^2 \, dx_2^2, \quad II = r_1 A_1 \, dx_1^2 + r_2 A_2 \, dx_2^2.
\]

**Proof.** We have proved that the compatibility condition for (3.3.3) is the Gauss-Codazzi equation (3.3.4), so by the Forbenius Theorem (3.3.3) is solvable. Let \( (e_1, e_2, e_3) \) be the solution with initial data

\[
(e_1, e_2, e_3)(p_0) = (v_1, v_2, v_3).
\]

Since \( P, Q \) are skew-symmetric and \( (v_1, v_2, v_3) \) is an orthogonal matrix, by Proposition 1.0.6 of the section on Frobenius Theorem that the solution \( (e_1, e_2, e_3)(p) \) is an orthogonal matrix for all \( p \in \mathcal{O} \). To construct the surface, we need to solve

\[
\begin{cases}
  f_{x_1} = A_1 e_1, \\
  f_{x_2} = A_2 e_2.
\end{cases}
\]

Note that the right hand side is known, so this system is solvable if and only if

\[
(A_1 e_1)_{x_2} = (A_2 e_2)_{x_1}.
\]

To see this, we compute

\[
(A_1 e_1)_{x_2} = (A_1)_{x_2} e_1 + A_1 (e_1)_{x_2} = (A_1)_{x_2} e_1 + A_1 \left( \frac{A_2}{A_1} \right)_{x_2} e_2
\]

\[
= (A_1)_{x_2} e_1 + (A_2)_{x_1} e_2,
\]

\[
(A_2 e_2)_{x_1} = (A_2)_{x_1} e_2 + A_2 (e_2)_{x_1} = (A_2)_{x_1} e_2 + A_2 \left( \frac{A_1}{A_2} \right)_{x_1} e_1
\]

\[
= (A_2)_{x_1} e_2 + (A_1)_{x_2} e_1,
\]

and see that \( (A_1 e_1)_{x_2} = (A_2 e_2)_{x_1} \), so (3.4.1) is solvable. Hence we can solve (3.4.1) by integration. It then follows that \( (f, e_1, e_2, e_3) \) is a solution of (3.3.5) with initial data \( (y_0, v_1, v_2, v_3) \). But (3.4.1) implies that \( f_{x_1}, f_{x_2} \) are linearly independent, so \( f \) is a parametrized surface, \( e_3 \) is normal to \( f \), and \( I = A_1^2 dx_1^2 + A_2^2 dx_2^2 \). Recall that

\[
\ell_{ij} = f_{x_i x_j} \cdot N = f_{x_i x_j} \cdot e_3 = -(e_3)_{x_i} \cdot e_j,
\]

So \( \ell_{11} = -(e_3)_{x_1} \cdot f_{x_1} = -(-r_1 e_1) \cdot A_1 e_1 = r_1, \ \ell_{12} = -(e_3)_{x_2} \cdot A_1 e_1 = 0, \) and \( \ell_{22} = -(e_3)_{x_2} \cdot f_{x_2} = -(-r_2 e_2) \cdot A_2 e_2 = r_2 A_2 \). Thus \( II = r_1 A_1 \, dx_1^2 + r_2 A_2 \, dx_2^2 \). □

**Corollary 3.4.2.** Suppose \( f, g : \mathcal{O} \to \mathbb{R}^3 \) are two surfaces parametrized by line of curvature coordinates, and \( f, g \) have the same first and second fundamental forms

\[
I = A_1^2 \, dx_1^2 + A_2^2 \, dx_2^2, \quad II = r_1 A_1 \, dx_1^2 + r_2 A_2 \, dx_2^2.
\]
Then there exists a rigid motion \( \phi \) of \( \mathbb{R}^3 \) such that \( g = \phi \circ f \).

Proof. Let

\[
e_1 = \frac{f_{x_1}}{A_1}, \quad e_2 = \frac{f_{x_2}}{A_2}, \quad e_3 = \frac{f_{x_1} \times f_{x_2}}{||f_{x_1} \times f_{x_2}||},
\]

\[
\xi_1 = \frac{g_{x_1}}{A_1}, \quad \xi_2 = \frac{g_{x_2}}{A_2}, \quad \xi_3 = \frac{g_{x_1} \times g_{x_2}}{||f_{x_1} \times f_{x_2}||},
\]

Fix \( p_0 \in \mathcal{O} \), and let \( \phi(x) = Tx + b \) be the rigid motion such that \( \phi(f(p_0)) = g(p_0) \) and \( T(e_i(p_0)) = \xi_i(p_0) \) for all \( 1 \leq i \leq 3 \). Then

1. \( \phi \circ f \) have the same I, II as \( f \), so \( \phi \circ f \) and \( g \) have the same I, II,
2. the o.n. moving frame for \( \phi \circ f \) is \( (Te_1, Te_2, Te_3) \).

Thus both \( (\phi \circ f, Te_1, Te_2, Te_3) \) and \( (g, \xi_1, \xi_2, \xi_3) \) are solutions of (3.3.5) with the same initial condition \( (g(p_0), \xi_1(p_0), \xi_2(p_0), \xi_3(p_0)) \). But Frobenius Theorem says that there is a unique solution for the initial value problem, hence

\[
(\phi \circ f, T\xi_1, T\xi_2, T\xi_3) = (g, \xi_1, \xi_2, \xi_3).
\]

In particular, this proves that \( \phi \circ f = g \). \( \square \)

### 3.5. Gauss Theorem in line of curvature coordinates.

We know that the Gaussian curvature \( K \) depends on both I and II. The Gauss Theorem says that in fact \( K \) can be computed from I alone. We will first prove this when the surface is parametrized by line of curvatures.

**Theorem 3.5.1. Gauss Theorem in line of curvature coordinates**

Suppose \( f : \mathcal{O} \to \mathbb{R}^3 \) is a surface parametrized by line of curvatures, and

\[
I = A_1^2 dx_1^2 + A_2^2 dx_2^2, \quad II = r_1 A_1 dx_1^2 + r_2 A_2 dx_2^2.
\]

Then

\[
K = -\frac{\frac{(A_1)_{x_2}}{A_2} x_2 + \frac{(A_2)_{x_1}}{A_1} x_1}{A_1 A_2},
\]

so \( K \) can be computed from I alone.

**Proof.** Recall that

\[
K = \frac{\det(\ell_{ij})}{\det(g_{ij})} = \frac{r_1 A_1 r_2 A_2}{A_1^2 A_2^2} = \frac{r_1 r_2}{A_1 A_2}.
\]

But the first equation in the Gauss-Codazzi equation is

\[
\left( \frac{(A_1)_{x_2}}{A_2} \right) x_2 + \left( \frac{(A_2)_{x_1}}{A_1} \right) x_1 = -r_1 r_2.
\]

So

\[
K = -\frac{\frac{(A_1)_{x_2}}{A_2} x_2 + \frac{(A_2)_{x_1}}{A_1} x_1}{A_1 A_2}.
\]

\( \square \)
3.6. Gauss-Codazzi equation in local coordinates.

We will derive the Gauss-Codazzi equations for arbitrary parametrized surface \( f : \mathcal{O} \to U \subset \mathbb{R}^3 \).

Our experience in curve theory tells us that we should find a moving frame on the surface and then differentiate the moving frame to get relations among the invariants. We will use moving frames \( F = (v_1, v_2, v_3) \) on the surface to derive the relations among local invariants, where \( v_1 = f_{x_1}, v_2 = f_{x_2}, \) and \( v_3 = N \) the unit normal. Express the \( x \) and \( y \) derivatives of the local frame \( v_i \) in terms of \( v_1, v_2, v_3 \), then their coefficients can be written in terms of the two fundamental forms. Since \( (v_i)_{xy} = (v_i)_{yx} \), we obtain a PDE relation for \( I \) and \( II \). This is the Gauss-Codazzi equation of the surface. Conversely, given two symmetric bilinear forms \( g, b \) on an open subset \( \mathcal{O} \) of \( \mathbb{R}^2 \) such that \( g \) is positive definite and \( g, b \) satisfies the Gauss-Codazzi equation, then by the Frobenius Theorem there exists a surface in \( \mathbb{R}^3 \) unique up to rigid motion having \( g, b \) as the first and second fundamental forms respectively.

We use the frame \( (f_{x_1}, f_{x_2}, N) \), where

\[
N = \frac{f_{x_1} \times f_{x_2}}{\| f_{x_1} \times f_{x_2} \|}
\]

is the unit normal vector field. Since \( f_{x_1}, f_{x_2}, N \) form a basis of \( \mathbb{R}^3 \), the partial derivatives of \( f_x \) and \( N \) can be written as linear combinations of \( f_{x_1}, f_{x_2} \) and \( N \). So we have

\[
\begin{align*}
(f_{x_1}, f_{x_2}, N)_{x_1} &= (f_{x_1}, f_{x_2}, N)P,
(f_{x_1}, f_{x_2}, N)_{x_2} &= (f_{x_1}, f_{x_2}, N)Q,
\end{align*}
\]

where \( P = (p_{ij}), Q = (q_{ij}) \) are \( \text{gl}(3) \)-valued maps. This means that

\[
\begin{align*}
f_{x_1 x_1} &= p_{11} f_{x_1} + p_{21} f_{x_2} + p_{31} N, \\
f_{x_1 x_2} &= p_{12} f_{x_1} + p_{22} f_{x_2} + p_{32} N, \\
f_{x_1 N} &= p_{13} f_{x_1} + p_{23} f_{x_2} + p_{33} N, \\
f_{x_2 x_1} &= q_{11} f_{x_1} + q_{21} f_{x_2} + q_{31} N, \\
f_{x_2 x_2} &= q_{12} f_{x_1} + q_{22} f_{x_2} + q_{32} N, \\
f_{x_2 N} &= q_{13} f_{x_1} + q_{23} f_{x_2} + q_{33} N.
\end{align*}
\]

Recall that the fundamental forms are given by

\[
g_{ij} = f_{x_i} \cdot f_{x_j}, \quad \ell_{ij} = -f_{x_i} \cdot N_{x_j} = f_{x_i x_j} \cdot N.
\]

We want to express \( P \) and \( Q \) in terms of \( g_{ij} \) and \( h_{ij} \). To do this, we need the following Propositions.

**Proposition 3.6.1.** Let \( V \) be a vector space with an inner product \((,\), \( v_1, \cdots, v_n \) a basis of \( V \), and \( g_{ij} = (v_i, v_j) \). Let \( \xi \in V \), \( \xi_i = (\xi, v_i) \), and \( \xi = \sum_{i=1}^n x_i v_i \). Then

\[
\begin{pmatrix}
  x_1 \\
  \vdots \\
  x_n
\end{pmatrix}
= G^{-1}
\begin{pmatrix}
  \xi_1 \\
  \vdots \\
  \xi_n
\end{pmatrix},
\]

where \( G = (g_{ij}) \).

**Proof.** Note that

\[
\xi_i = (\xi, v_i) = \sum_{j=1}^n x_j v_j, v_i = \sum_{j=1}^n x_j (v_j, v_i) = \sum_{j=1}^n g_{ji} x_j.
\]
So \((\xi_1, \cdots, \xi_n)^4 = G(x_1, \cdots, x_n)^4\).

\[\Box\]

**Proposition 3.6.2.** The following statements are true:

1. The \(gl(3)\) valued functions \(P = (p_{ij})\) and \(Q = (q_{ij})\) in equation (3.6.1) can be written in terms of \(g_{ij}, \ell_{ij}\), and first partial derivatives of \(g_{ij}\).

2. The entries \(\{p_{ij}, q_{ij} \mid 1 \leq i, j \leq 2\}\) can be computed from the first fundamental form.

**Proof.** We claim that

\[f_{x_i x_j} \cdot f_{x_k}, \quad f_{x_i x_j} \cdot N, \quad N_{x_i} \cdot f_{x_j}, \quad N_{x_i} \cdot N,\]

can be expressed in terms of \(g_{ij}, \ell_{ij}\) and first partial derivatives of \(g_{ij}\). Then the Proposition follows from Proposition 3.6.1. To prove the claim, we proceed as follows:

\[
\begin{cases}
    f_{x_i x_i} \cdot f_{x_i} = \frac{1}{2}(g_{ii})_{x_i}, \\
    f_{x_i x_j} \cdot f_{x_i} = \frac{1}{2}(g_{ii})_{x_j}, & \text{if } i \neq j, \\
    f_{x_i x_i} \cdot f_{x_j} = (f_{x_i} \cdot f_{x_j})_{x_i} - f_{x_i} \cdot f_{x_i x_j} = (g_{ij})_{x_i} - \frac{1}{2}(g_{ii})_{x_j}, & \text{if } i \neq j
\end{cases}
\]

\[
\begin{cases}
    f_{x_i x_j} \cdot N = \ell_{ij}, \\
    N_{x_i} \cdot f_{x_j} = -\ell_{ij}, \\
    N_{x_i} \cdot N = 0.
\end{cases}
\]

Let

\[G = \begin{pmatrix} g_{11} & g_{12} & 0 \\ g_{12} & g_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

By Proposition 3.6.1, we have

\[
P = \left( \frac{1}{2}(g_{11})_{x_1} \begin{pmatrix} g_{11} & g_{12} & 0 \\ g_{12} & g_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1/2(g_{11})_{x_1} & 1/2(g_{11})_{x_2} & -\ell_{11} \\ 1/2(g_{12})_{x_1} & 1/2(g_{22})_{x_1} & -\ell_{12} \\ \ell_{11} & \ell_{12} & 0 \end{pmatrix}
\]

\[= G^{-1}A_1,\]

\[
Q = \begin{pmatrix} g_{11} & g_{12} & 0 \\ g_{12} & g_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1/2(g_{11})_{x_2} & (g_{12})_{x_2} - 1/2(g_{22})_{x_1} & -\ell_{12} \\ 1/2(g_{12})_{x_2} & (g_{22})_{x_2} - 1/2(g_{11})_{x_1} & -\ell_{22} \\ \ell_{12} & \ell_{22} & 0 \end{pmatrix}
\]

\[= G^{-1}A_2.
\]

This proves the Proposition.

Formula (3.6.2) gives explicit formulas for entries of \(P\) and \(Q\) in terms of \(g_{ij}\) and \(\ell_{ij}\). Moreover, they are related to the Christoffel symbols \(\Gamma^i_{jk}\) arise in the geodesic equation (3.3.1) in Theorem 3.3. Recall that

\[
\Gamma^i_{jk} = \frac{1}{2}g^{km}[ij, m],
\]

where \((g^{ij})\) is the inverse matrix of \((g_{ij})\), \([ij, k] = g_{ki,j} + g_{jk,i} - g_{ij,k}\), and \(g_{ij,k} = \frac{\partial g_{ij}}{\partial x_k}\).

**Theorem 3.6.3.** For \(1 \leq i, j \leq 2\), we have

\[
(3.6.3) \quad p_{ji} = \Gamma^j_{i1}, \quad q_{ji} = \Gamma^j_{i2}
\]
Proof. Note that (3.6.4) implies

\[
\begin{align*}
\frac{p_{11}}{2} g^{11} g_{11,1} + g^{12} (g_{12,1} - \frac{1}{2} g_{11,2}) &= \Gamma_{11}^1, \\
\frac{p_{12}}{2} g^{11} g_{11,2} + \frac{1}{2} g^{12} g_{22,1} &= \Gamma_{21}^1, \\
\frac{p_{21}}{2} g^{12} g_{11,1} + g^{22} (g_{12,1} - \frac{1}{2} g_{11,2}) &= \Gamma_{11}^2, \\
\frac{p_{22}}{2} g^{12} g_{11,2} + \frac{1}{2} g^{22} g_{22,1} &= \Gamma_{21}^2, \\
\frac{q_{11}}{2} g^{11} g_{11,2} + \frac{1}{2} g^{12} g_{22,1} &= \Gamma_{12}^1, \\
\frac{q_{12}}{2} g^{11} g_{12,2} - \frac{1}{2} g^{22} g_{22,2} &= \Gamma_{11}^2, \\
\frac{q_{21}}{2} g^{12} g_{11,2} + \frac{1}{2} g^{22} g_{22,1} &= \Gamma_{12}^2, \\
\frac{q_{22}}{2} g^{12} g_{12,2} - \frac{1}{2} g^{22} g_{22,2} &= \Gamma_{22}^2.
\end{align*}
\]

(3.6.4)

Note that

\[ q_{11} = p_{12}, \quad q_{21} = p_{22}. \]

Theorem 3.6.4. The Fundamental Theorem of surfaces in \( \mathbb{R}^3 \).

Suppose \( f : \mathcal{O} \to \mathbb{R}^3 \) is a parametrized surface, and \( g_{ij}, \ell_{ij} \) are the coefficients of I,II. Let \( P, Q \) be the smooth \( \text{gl}(3) \)-valued maps defined in terms of \( g_{ij} \) and \( \ell_{ij} \) by (3.6.3). Then \( P, Q \) satisfy

\[
P_{x_2} - Q_{x_1} = [P, Q].
\]

Conversely, let \( \mathcal{O} \) be an open subset of \( \mathbb{R}^2 \), \( (g_{ij}, \ell_{ij}) : \mathcal{O} \to \text{gl}(2) \) smooth maps such that \( g_{ij} \) is positive definite and \( \ell_{ij} \) is symmetric, and \( P, Q : U \to \text{gl}(3) \) the maps defined by (3.6.2). Suppose \( P, Q \) satisfy the compatibility equation (3.6.5). Let \( (x^0_1, x^0_2) \in \mathcal{O}, \ p_0 \in \mathbb{R}^3 \), and \( u_1, u_2, u_3 \) a basis of \( \mathbb{R}^3 \) so that \( u_i \cdot u_j = g_{ij}(x^0_1, x^0_2) \) and \( u_i \cdot u_3 = 0 \) for \( 1 \leq i, j \leq 2 \). Then there exists an open subset \( \mathcal{O}_0 \subset \mathcal{O} \) of \( (x^0_1, x^0_2) \) and a unique immersion \( f : \mathcal{O}_0 \to \mathbb{R}^3 \) so that \( f \) maps \( \mathcal{O}_0 \) homeomorphically to \( f(\mathcal{O}_0) \) such that

1. the first and second fundamental forms of the embedded surface \( f(\mathcal{O}_0) \) are given by \( (g_{ij}) \) and \( (\ell_{ij}) \) respectively,
2. \( f(x^0_1, x^0_2) = p_0 \), and \( f_x(x^0_1, x^0_2) = u_i \) for \( i = 1, 2 \).

Proof. We have proved the first half of the theorem, and it remains to prove the second half. We assume that \( P, Q \) satisfy the compatibility condition (3.6.5). So Frobenius Theorem ?? implies that the following system has a unique local solution

\[
\begin{align*}
(v_1, v_2, v_3)_{x_1} &= (v_1, v_2, v_3)P, \\
(v_1, v_2, v_3)_{x_2} &= (v_1, v_2, v_3)Q, \\
(v_1, v_2, v_3)(x^0_1, x^0_2) &= (u_1, u_2, u_3).
\end{align*}
\]

(3.6.6)

Next we want to solve

\[
\begin{align*}
f_{x_1} &= v_1, \\
f_{x_2} &= v_2, \\
f(x^0_1, x^0_2) &= p_0.
\end{align*}
\]

The compatibility condition is \( (v_1)_{x_2} = (v_2)_{x_1} \). But

\[
(v_1)_{x_2} = \sum_{j=1}^3 q_{j1} v_j, \quad (v_2)_{x_1} = \sum_{j=1}^3 p_{j2} v_j.
\]
It follows from (3.6.2) that the second column of $P$ is equal to the first column of $Q$. So $(v_1)_{x_2} = (v_2)_{x_1}$, and hence there exists a unique $f$.

We will prove below that $f$ is an immersion, $v_3$ is perpendicular to the surface $f$, $||v_3|| = 1$, $f_{x_i} \cdot f_{x_j} = g_{ij}$, and $(v_3)_{x_i} \cdot f_{x_j} = -\ell_{ij}$, i.e., $f$ is a surface in $\mathbb{R}^3$ with

$$
I = \sum_{ij} g_{ij} dx_i dx_j, \quad II = \sum_{ij} \ell_{ij} dx_i dx_j
$$

as its first and second fundamental forms. The first step is to prove that the $3 \times 3$ matrix function $\Phi = (v_i \cdot v_j)$ is equal to the matrix $G$ defined by (3.6.2). To do this, we compute the first derivative of $\Phi$. Since $v_1, v_2, v_3$ satisfy (3.6.6), a direct computation gives

$$(v_i \cdot v_j)_{x_1} = (v_i)_{x_1} \cdot v_j + v_i \cdot (v_j)_{x_1},$$

$$= \sum_k p_k i v_k \cdot v_j + p_k j v_k \cdot v_i = \sum_k p_k i g_{jk} + g_{ik} p_{kj},$$

$$= (GP)_{j} + (GP)_{ij} = (GP + (GP)^t)_{ij}.$$  

Formula (3.6.2) implies that $GP = G(G^{-1} A_1) = A_1$ and $A_1 + A_1^t = G_{x_1}$. Hence $(GP)^t + GP = G_{x_1}$ and

$$\Phi_{x_1} = G_{x_1}.$$  

A similar computation implies that

$$\Phi_{x_2} = G_{x_2}.$$  

But the initial value $\Phi(x_1^0, x_2^0) = G(x_1^0, x_2^0)$. So $\Phi = G$. In other words, we have shown that

$$f_{x_i} \cdot f_{x_j} = g_{ij}, \quad f_{x_i} \cdot v_3 = 0.$$  

Thus

1. $f_{x_1}, f_{x_2}$ are linearly independent, i.e., $f$ is an immersion,
2. $v_3$ is the unit normal field to the surface $f$,
3. the first fundamental form of $f$ is $\sum_{ij} g_{ij} dx_i dx_j$.

To compute the second fundamental form of $f$, we use (3.6.6) to compute

$$-(v_3)_{x_1} \cdot v_j = (g^{11} \ell_{11} + g^{12} \ell_{12}) v_1 \cdot v_j + (g^{12} \ell_{11} + g^{22} \ell_{12}) v_2 \cdot v_j$$

$$= (g^{11} \ell_{11} + g^{12} \ell_{12}) g_{1j} + (g^{12} \ell_{11} + g^{22} \ell_{12}) g_{2j}$$

$$= \ell_{11} (g^{11} g_{1j} + g^{12} g_{2j}) + \ell_{12} (g^{21} g_{1j} + g^{22} g_{2j})$$

$$= \ell_{11} \delta_{1j} + \ell_{12} \delta_{2j}.$$  

So

$$-(v_3)_{x_1} \cdot v_1 = \ell_{11}, \quad -(v_3)_{x_1} \cdot v_2 = \ell_{12}.$$  

Similar computations imply that

$$-(v_2)_{x_2} \cdot v_j = \ell_{2j}.$$  

This proves that $\sum_{ij} \ell_{ij} dx_i dx_j$ is the second fundamental form of $f$. \qed

System (3.6.5) with $P, Q$ defined by (3.6.2) is called the **Gauss-Codazzi equation for the surface $f(O)$**, which is a second order PDE with 9 equations for six functions $g_{ij}$ and $\ell_{ij}$. Equation (3.6.5) is too complicated to memorize. It is more useful and simpler to just remember how to derive the Gauss-Codazzi equation.

It follows from (3.6.1), (3.6.2), and (3.6.3) that we have
\[ f_{x,x_1} = \sum_{j=1}^{2} p_{ij} f_{x_j} + \ell_{i1} N = \sum_{j=1}^{2} \Gamma_{i1}^j f_{x_j} + \ell_{i1} N, \]
\[ f_{x,x_2} = \sum_{j=1}^{2} q_{j2} f_{x_j} + \ell_{i2} N = \sum_{j=1}^{2} \Gamma_{i2}^j f_{x_j} + \ell_{i2} N, \]

where \( p_{ij} \) and \( q_{ij} \) are defined in (3.6.2).

So we have

\[
(3.6.7) \quad f_{x,x_j} = \Gamma_{i1}^j f_{x_1} + \Gamma_{i2}^j f_{x_2} + \ell_{ij} N,
\]

**Proposition 3.6.5.** Let \( f : \mathcal{O} \to \mathbb{R}^3 \) be a local coordinate system of an embedded surface \( M \) in \( \mathbb{R}^3 \), and \( \alpha(t) = f(x_1(t), x_2(t)) \). Then \( \alpha \) satisfies the geodesic equation (3.6.5) if and only if \( \alpha''(t) \) is normal to \( M \) at \( \alpha(t) \) for all \( t \).

**Proof.** Differentiate \( \alpha' \) to get \( \alpha' = \sum_{i=1}^{2} f_{x_i} x'_i \). So

\[
\alpha'' = \sum_{i,j=1}^{2} f_{x_i,x_j} x'_i x'_j + f_{x_i} x''_i
\]
\[
= \sum_{i,j,k=1}^{2} \Gamma_{ij}^k f_{x_k} x'_i x'_j + \ell_{ij} N + f_{x_i} x''_i
\]
\[
= \sum_{i,j=1}^{2} (\Gamma_{ij}^k x'_i x'_j + x''_i) f_{x_k} + \ell_{ij} N = 0 + \ell_{ij} N = \ell_{ij} N.
\]

\[ \square \]

### 3.7. The Gauss Theorem.

Equation (3.6.5) is the *Gauss-Codazzi equation* for \( M \).

The Gaussian curvature \( K \) is defined to be the determinant of the shape operator \(-dN\), which depends on both the first and second fundamental forms of the surface. In fact, by Proposition 3.6.2,

\[
K = \frac{\ell_{11} \ell_{22} - \ell_{12}^2}{g_{11} g_{22} - g_{12}^2}.
\]

We will show below that \( K \) can be computed in terms of \( g_{ij} \) alone. Equate the 12 entry of equation (3.6.5) to get

\[
(p_{12})_{x_2} - (q_{12})_{x_1} = \sum_{j=1}^{3} p_{1j} q_{j2} - q_{1j} p_{j2}.
\]

Recall that formula (3.6.4) gives \( \{p_{ij}, q_{ij} \mid 1 \leq i, j \leq 2\} \) in terms of the first fundamental form \( I \). We move terms involves \( p_{ij}, q_{ij} \) with \( 1 \leq i, j \leq 2 \) to one side
to get

\[(3.7.1) \quad (p_{12})_{x_2} - (q_{12})_{x_1} - \sum_{j=1}^{2} p_{1j} q_{j2} - q_{1j} p_{j2} = p_{13} q_{32} - q_{13} p_{32}.\]

We claim that the right hand side of (3.7.1) is equal to

\[-g^{11}(\ell_{11}\ell_{22} - \ell_{12}^2) = -g^{11}(g_{11} g_{22} - g_{12}^2) K.\]

To prove this claim, use (3.6.2) to compute $P, Q$ to get

\[p_{13} = -(g^{11} \ell_{11} + g^{12} \ell_{12}), \quad p_{32} = \ell_{12},\]
\[q_{13} = -(g^{11} \ell_{12} + g^{12} \ell_{22}), \quad q_{32} = \ell_{22}.\]

So we get

\[(3.7.2) \quad (p_{12})_{x_2} - (q_{12})_{x_1} - \sum_{j=1}^{2} p_{1j} q_{j2} - q_{1j} p_{j2} = -g^{11}(g_{11} g_{22} - g_{12}^2) K.\]

Hence we have proved the claim and also obtained a formula of $K$ purely in terms of $g_{ij}$ and their derivatives:

\[K = -\frac{(p_{12})_{x_2} - (q_{12})_{x_1} - \sum_{j=1}^{2} p_{1j} q_{j2} - q_{1j} p_{j2}}{g^{11}(g_{11} g_{22} - g_{12}^2)}.\]

This proves

**Theorem 3.7.1. Gauss Theorem.** The Gaussian curvature of a surface in $\mathbb{R}^3$ can be computed from the first fundamental form.

The equation (3.7.2), obtained by equating the 12-entry of (3.6.5), is the **Gauss equation**.

A geometric quantity on an embedded surface $M$ in $\mathbb{R}^n$ is called **intrinsic** if it only depends on the first fundamental form $I$. Otherwise, the property is called **extrinsic**, i.e., it depends on both $I$ and $II$.

We have seen that the Gaussian curvature and geodesics are intrinsic quantities, and the mean curvature is extrinsic.

If $\phi : M_1 \rightarrow M_2$ is a diffeomorphism and $f(x_1, x_2)$ is a local coordinates on $M_1$, then $\phi \circ f(x_1, x_2)$ is a local coordinate system of $M_2$. The diffeomorphism $\phi$ is an **isometry** if the first fundamental forms for $M_1, M_2$ are the same written in terms of $dx_1, dx_2$. In particular,

(i) $\phi$ preserves angles and arc length, i.e., the arc length of the curve $\phi(\alpha)$ is the same as the curve $\alpha$ and the angle between the curves $\phi(\alpha)$ and $\phi(\beta)$ is the same as the angle between $\alpha$ and $\beta$,

(ii) $\phi$ maps geodesics to geodesics.

Euclidean plane geometry studies the geometry of triangles. Note that triangles can be viewed as a triangle in the plane with each side being a geodesic. So a natural definition of a triangle on an embedded surface $M$ is a piecewise smooth curve with three geodesic sides and any two sides meet at an angle lie in $(0, \pi)$. One important problem in geometric theory of $M$ is to understand the geometry of triangles on $M$. For example, what is the sum of interior angles of a triangle on an embedded surface $M$? This will be answered by the Gauss-Bonnet Theorem.
Note that the first fundamental forms for the plane
\[ f(x_1, x_2) = (x_1, x_2, 0) \]
and the cylinder
\[ h(x_1, x_2) = (\cos x_1, \sin x_1, x_2) \]
have the same and is equal to \( I = dx_1^2 + dx_2^2 \), and both surfaces have constant zero Gaussian curvature (cf. Examples ?? and ??). We have also proved that geodesics are determined by \( I \) alone. So the geometry of triangles on the cylinder is the same as the geometry of triangles in the plane. For example, the sum of interior angles of a triangle on the plane (and hence on the cylinder) must be \( \pi \). In fact, let \( \phi \) denote the map from \( (0, 2\pi) \times \mathbb{R} \) to the cylinder minus the line \((1, 0, x_2)\) defined by
\[ \phi(x_1, x_2, 0) = (\cos x_1, \sin x_1, x_2). \]
Then \( \phi \) is an isometry.

3.8. Gauss-Codazzi equation in orthogonal coordinates.

If the local coordinates \( x_1, x_2 \) are orthogonal, i.e., \( g_{12} = 0 \), then the Gauss-Codazzi equation (3.6.5) becomes much simpler. Instead of putting \( g_{12} = 0 \) to (3.6.5), we derive the Gauss-Codazzi equation directly using an o.n. moving frame.

We write
\[ g_{11} = A_1^2, \quad g_{22} = A_2^2, \quad g_{12} = 0. \]

Let
\[ e_1 = \frac{f_{x_1}}{A_1}, \quad e_2 = \frac{f_{x_2}}{A_2}, \quad e_3 = N. \]

Then \( (e_1, e_2, e_3) \) is an o.n. moving frame on \( M \). Write
\[ \left\{ (e_1, e_2, e_3)_{x_1} = (e_1, e_2, e_3) \tilde{P}, \right. \]
\[ \left. (e_1, e_2, e_3)_{x_2} = (e_1, e_2, e_3) \tilde{Q}. \right. \]

Since \( (e_1, e_2, e_3) \) is orthogonal, \( \tilde{P}, \tilde{Q} \) are skew-symmetric. Moreover,
\[ \tilde{p}_{ij} = (e_j)_{x_i} \cdot e_i, \quad \tilde{q}_{ij} = (e_j)_{x_2} \cdot e_i. \]

A direct computation gives
\[ (e_1)_{x_1} \cdot e_2 = \frac{f_{x_1}}{A_1} \frac{f_{x_2}}{A_2} - \frac{f_{x_1 x_2} \cdot f_{x_1}}{A_1 A_2} = \frac{(f_{x_1} \cdot f_{x_2})_{x_1} - f_{x_1} \cdot f_{x_1 x_2}}{A_1 A_2} = -\frac{(\frac{1}{2} A_1^2)_{x_2}}{A_1 A_2} = -\frac{(A_1)_{x_2}}{A_2}. \]

Similar computation gives the coefficients \( \tilde{p}_{ij} \) and \( \tilde{q}_{ij} \):

\[ (3.8.1) \quad \tilde{P} = \begin{pmatrix} 0 & \frac{(A_1)_{x_2}}{A_2} & -\frac{\ell_{11}}{A_1} \\ \frac{(A_1)_{x_2}}{A_2} & 0 & -\frac{\ell_{12}}{A_2} \\ -\frac{\ell_{11}}{A_1} & -\frac{\ell_{12}}{A_2} & 0 \end{pmatrix}, \quad \tilde{Q} = \begin{pmatrix} 0 & -\frac{(A_2)_{x_1}}{A_1} & -\frac{\ell_{12}}{A_1} \\ -\frac{(A_2)_{x_1}}{A_1} & 0 & -\frac{\ell_{22}}{A_2} \\ -\frac{\ell_{12}}{A_1} & -\frac{\ell_{22}}{A_2} & 0 \end{pmatrix}. \]
To get the Gauss-Codazzi equation of the surface parametrized by an orthogonal coordinates we only need to compute the 21-th, 31-th, and 32-the entry of the following equation

\[(\bar{P})_{x_2} - (\bar{Q})_{x_1} = [\bar{P}, \bar{Q}],\]

and we obtain

\[
\begin{cases}
- \left(\frac{(A_1)_{x_2}}{A_2}\right)_{x_2} - \left(\frac{(A_2)_{x_1}}{A_1}\right)_{x_1} = \ell_{11} \ell_{22} - \ell_{12}^2, \\
\left(\frac{\ell_{11}}{A_1}\right)_{x_2} - \left(\frac{\ell_{12}}{A_1}\right)_{x_1} = \frac{\ell_{12}(A_2)_{x_1}}{A_1 A_2} + \frac{\ell_{22}(A_1)_{x_2}}{A_2^2}, \\
\left(\frac{\ell_{12}}{A_2}\right)_{x_2} - \left(\frac{\ell_{22}}{A_2}\right)_{x_1} = -\frac{\ell_{11}(A_2)_{x_1}}{A_2^2} - \frac{\ell_{12}(A_1)_{x_2}}{A_1 A_2}.
\end{cases}
\]

The first equation of (3.8.2) is called the Gauss equation. Note that the Gaussian curvature is

\[K = \frac{\ell_{11} \ell_{22} - \ell_{12}^2}{(A_1 A_2)^2}.\]

So we have

\[
K = -\left(\frac{(A_1)_{x_2}}{A_2}\right)_{x_2} + \left(\frac{(A_2)_{x_1}}{A_1}\right)_{x_1}.
\]

We have seen that the Gauss-Codazzi equation becomes much simpler in orthogonal coordinates. Can we always find local orthogonal coordinates on a surface in \(\mathbb{R}^3\)? This question can be answered by the following theorem, which we state without a proof.

**Theorem 3.8.1.** Suppose \(f : \mathcal{O} \to \mathbb{R}^3\) be a surface, \(x_0 \in \mathcal{O}\), and \(Y_1, Y_2 : \mathcal{O} \to \mathbb{R}^3\) smooth maps so that \(Y_1(x_0), Y_2(x_0)\) are linearly independent and tangent to \(M = f(\mathcal{O})\) at \(f(x_0)\). Then there exist open subset \(\mathcal{O}_0\) of \(\mathcal{O}\) containing \(x_0\), open subset \(\mathcal{O}_1\) of \(\mathbb{R}^2\), and a diffeomorphism \(h : \mathcal{O}_1 \to \mathcal{O}_0\) so that \((f \circ h)_{y_1}\) and \((f \circ h)_{y_2}\) are parallel to \(Y_1 \circ h\) and \(Y_2 \circ h\), respectively.

The above theorem says that if we have two linearly independent vector fields \(Y_1, Y_2\) on a surface, then we can find a local coordinate system \(\phi(y_1, y_2)\) so that \(\phi_{y_1}, \phi_{y_2}\) are parallel to \(Y_1, Y_2\) respectively.

Given an arbitrary local coordinate system \(f(x_1, x_2)\) on \(M\), we apply the Gram-Schmidt process to \(f_{x_1}, f_{x_2}\) to construct smooth o.n. vector fields \(e_1, e_2:\)

\[
e_1 = \frac{f_{x_1}}{\sqrt{g_{11}}},
\]

\[
e_2 = \frac{\sqrt{g_{11}}(f_{x_2} - \frac{g_{12}}{g_{11}} f_{x_1})}{\sqrt{g_{11}g_{22} - g_{12}^2}}.
\]

By Theorem 3.8.1, there exists new local coordinate system \(\tilde{f}(y_1, y_2)\) so that \(\frac{\partial f}{\partial x_1}\) and \(\frac{\partial f}{\partial x_2}\) are parallel to \(e_1\) and \(e_2\). So the first fundamental form written in this coordinate system has the form

\[
\tilde{g}_{11} dy_1^2 + \tilde{g}_{22} dy_2^2.
\]

However, in general we can not find coordinate system \(\tilde{f}(y_1, y_2)\) so that \(e_1\) and \(e_2\) are coordinate vector fields \(\frac{\partial f}{\partial y_1}\) and \(\frac{\partial f}{\partial y_2}\) because if we can then the first fundamental form of the surface is \(I = dy_1^2 + dy_2^2\), which implies that the Gaussian curvature of the surface must be zero.